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LAMINAR NATURAL-CONVECTIVE HEAT TRANSFER FROM THE OUTER SURFACE OF A VERTICAL CYLINDER

TETSU FUJII and HARUO UEHARA

Institute of Industrial Science, Kyushu University, Fukuoka, Japan

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Abstract—Laminar natural convection along the outer surface of a vertical cylinder is compared with that along a vertical flat plate on heat transfer. For any Prandtl number and for arbitrary vertical temperature or heat flux distribution at the cylinder surface, local heat-transfer coefficients are represented non-dimensionally by the following approximate formulae.

When surface temperature distributions are given,

$$(Nu_x)_c = (Nu_x)_p + 0{\cdot}435\frac{x}{r_w}, \quad \frac{x}{r_w} \lesssim 0{\cdot}7\,(Nu_x)_p,$$

and when surface heat flux distributions are given,

$$(Nu_x)_c = (Nu_x)_p + 0{\cdot}345\frac{x}{r_w}, \quad \frac{x}{r_w} \lesssim 0{\cdot}7\,(Nu_x)_p,$$

where $(Nu_x)_c$ and $(Nu_x)_p$ are the local Nusselt numbers for a cylinder and a flat plate respectively, x the vertical distance from the leading edge, and r_w the radius of the cylinder.

NOMENCLATURE

$f(\eta, \xi), f^*(\eta^*, \xi^*)$, non-dimensional stream functions defined in (29) and (29)′ respectively;

$f_0(\eta), f_0^*(\eta^*)$, basic stream functions of perturbation or stream functions of similarity transformation for a flat plate, defined in (36) and (36)′ respectively;

$f_1(\eta), f_1^*(\eta^*)$, stream functions of the first perturbation, defined in (36) and (36)′ respectively;

$f_2(\eta), f_2^*(\eta^*)$, stream functions of the second perturbation, defined in (36) and (36)′ respectively;

Gr_x, local Grashof number defined by (9);

Gr_x^*, modified local Grashof number defined by (14);

g, gravitational acceleration;

$H(0)$, thermal boundary value defined by (44);

M, N, arbitrary constants in (10) and (5) respectively;

m, n, arbitrary exponents in (10) and (5) respectively;

Nu_r, Nusselt number defined by (43);

Nu_x, $\alpha_x x/\lambda$, local Nusselt number;

Pr, $\nu/\varkappa$, Prandtl number;

q_w, local heat flux at the heated surface;

r, radial distance from the cylinder axis;

r_w, radius of a cylinder;

t, temperature;

u, v, velocity components in x- and r-directions respectively;

x, vertical distance from the leading edge of the heated surface.

Greek symbols

α, $q_w/(t_w - t_\infty)$, heat-transfer coefficient;

β, average volumetric thermal expansion coefficient;

δ, thickness of the thermal boundary layer along a flat plate, defined by (15);

η, η^*, independent variables defined by (31) and (31)′ respectively;

$\theta(\eta, \xi), \theta^*(\eta^*, \xi^*)$, non-dimensional temperature profiles in the boundary layer, defined by (30) and (30)′ respectively.

$\theta_0(\eta), \theta_0^*(\eta^*)$, basic temperature profiles of perturbation or temperature profiles of similarity transformation for a flat plate, defined in (37) and (37)′ respectively;

$\theta_1(\eta), \theta_1^*(\eta^*)$, temperature profiles of the first perturbation, defined in (37) and (37)′ respectively;

$\theta_2(\eta), \theta_2^*(\eta^*)$, temperature profiles of the second perturbation, defined in (37) and (37)′ respectively;

$\varkappa$, thermal diffusivity;

λ, thermal conductivity;

ν, kinematic viscosity;

ξ, ξ^*, perturbation parameters defined by (7) and (12) respectively;

ψ, stream function defined in (28).

Superscripts

$*$, for the case of given surface heat flux;

$'$, differentiation with respect to η or η^*.

Subscripts

c, for a cylinder;

p, for a flat plate;

x, local values at x;

w, conditions at the heated surface;

∞, conditions in the ambient fluid.

1. INTRODUCTION

On the study of laminar natural convection along a vertical surface, theoretical analysis for a flat plate is simple and exact. In case of its experimental verification, however, it is desirable to use a vertical cylinder, in order to idealize the experimental conditions and make the measurements accurate. In this case, the radius of the cylinder must be sufficiently large, because the smaller is the radius, the larger is the heat transfer coefficient in comparison with that for a flat plate.

A number of solutions on the natural-convection boundary layer along a vertical cylinder were reported, by Sparrow and Gregg [1] and Hara [2] for uniform surface temperature and Prandtl number of 0·72, 0·733 and 1, by Mabuchi [3] and Fujii *et al.* [4] for uniform surface heat flux and Prandtl number of 0·72, 1, 5, 10 and 100, and by Millsaps and Pohlhausen [5] for uniform heat-transfer coefficient and Prandtl number of 0·733, 1, 10 and 100. Such simple thermal conditions as above appear seldom in practice. Especially for the case of uniform heat-transfer coefficient, it may be valuable only that similarity solutions are obtainable.

In this paper, it is attempted to reduce the formula for calculating the coefficient of heat transfer from the outer surface of a vertical cylinder with arbitrary vertical temperature or heat flux distribution to fluid of any Prandtl number, when the coefficient of heat transfer from a flat plate with the same thermal condition is known.

2. NUMERICAL SOLUTIONS AND CONSIDERATIONS

The pertinent boundary layer equations of continuity, motion and energy are respectively,

$$\frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial r} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = g\beta(t - t_\infty) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{2}$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial r} = \frac{\varkappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right). \tag{3}$$

These equations must be solved under the following boundary conditions.

$$\left.\begin{array}{ll} u = 0, \quad v = 0, \quad t = t_w \text{ or } -\lambda\dfrac{\partial t}{\partial r} = q_w, & \\ \qquad\qquad\qquad\qquad \text{at} \quad r = r_w, & \\ u = 0, \quad t = t_\infty, \qquad\quad \text{at} \quad r = \infty, & \end{array}\right\} \tag{4}$$

where t_w or q_w is an arbitrary function of x.

When t_w or q_w is given by a power function of x as shown in (5) or (10), these problems can be solved by perturbation method. Since the principle of this method is well known, the ordinary differential equations reduced are described in Appendix 1.

(i) *Cases of given surface temperature distributions*

Surface temperature distribution t_w is assumed as

$$t_w - t_\infty = Nx^n, \tag{5}$$

where t_∞ is uniform ambient fluid temperature, and N, n are arbitrary positive constants. The local Nusselt number is reduced from temperature profile (30), (37) as

$$\frac{(Nu_x)_c}{(Nu_x)_p} = 1 + \frac{\theta_1'(0)}{\theta_0'(0)}\xi + \frac{\theta_2'(0)}{\theta_0'(0)}\xi^2 + \ldots, \tag{6}$$

where perturbation parameter ξ, local Nusselt number for a flat plate $(Nu_x)_p$ and local Grashof number Gr_x are defined respectively by,

$$\xi = 2(\sqrt{2})\frac{x}{r_w}Gr_x^{-\frac{1}{4}}, \tag{7}$$

$$(Nu_x)_p = \frac{(\alpha_x)_p x}{\lambda} = \frac{-\theta_0'(0)}{\sqrt{2}}Gr_x^{\frac{1}{4}}, \tag{8}$$

$$Gr_x = \frac{g\beta(t_w - t_\infty)x^3}{\nu^2} = \frac{g\beta N x^{n+3}}{\nu^2}, \tag{9}$$

and $\theta_0'(0)$ is a thermal boundary value of equations (38), which are the basic equations of perturbation for a cylinder and are also reduced from similarity transformation of the boundary-layer equations for a flat plate, and $\theta_1'(0)$, $\theta_2'(0)$ are thermal boundary values of the first and second perturbation equations (39), (40) respectively. Some examples of these boundary values computed with an electronic computer, are shown in Table 1.

(ii) *Cases of given surface heat flux distributions*

Surface heat flux distribution q_w is assumed as

$$q_w = Mx^m, \tag{10}$$

where M and m are arbitrary positive constants. The local Nusselt number is reduced from temperature profile (30)′, (37)′ as

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \left[1 + \frac{\theta_1^*(0)}{\theta_0^*(0)}\xi^* + \frac{\theta_2^*(0)}{\theta_0^*(0)}\xi^{*2} + \ldots\right]^{-1}, \tag{11}$$

where perturbation parameter ξ^*, local Nusselt number for a flat plate $(Nu_x)_p$ and modified local Grashof number Gr_x^* are defined respectively by,

$$\xi^* = 2 \times 5^{\frac{1}{5}}\frac{x}{r_w}Gr_x^{*-\frac{1}{5}}, \tag{12}$$

$$(Nu_x)_p = \frac{1}{-5^{\frac{1}{5}}\theta_0^*(0)}Gr_x^{*\frac{1}{5}}, \tag{13}$$

$$Gr_x^* = \frac{g\beta q_w x^4}{\lambda\nu^2} = \frac{g\beta M x^{4+m}}{\lambda\nu^2}. \tag{14}$$

Some thermal boundary values $\theta_0^*(0)$, $\theta_1^*(0)$ and $\theta_2^*(0)$ for the case of $m = 0$ are shown in Table 2 [4].

(iii) *Transformation by boundary-layer thickness*

Table 1. Thermal boundary values for the cases of given surface temperature distribution

n	Pr	$\theta_0'(0)$	$\theta_1'(0)$	$\theta_2'(0)$	Symbols in Fig. 1
0	1	−0·5671	−0·2236	0·0250	●
0	100	−2·1910	−0·2254	0·0132	♦
0·1	1	−0·6093	−0·2255	0·0253	◉

In this paper, thickness of the thermal boundary layer along a flat plate δ is introduced as

$$\delta = \frac{x}{(Nu_x)_p}. \tag{15}$$

This is reduced when the hypothetical temperature profile in the boundary layer is linearly approximated such that the gradient coincides with the actual gradient at the heated surface. By substituting (8), (7) and (13), (12) into (15), the following relations are obtained respectively,

$$\frac{\delta}{r_w} = \frac{\xi}{-2\theta_0'(0)}, \tag{16}$$

$$\frac{\delta}{r_w} = \frac{-\theta_0^*(0)\,\xi^*}{2}. \tag{17}$$

Figure 1 shows a plot of $(Nu_x)_c/(Nu_x)_p$ vs. δ/r_w. Each term is evaluated by (6), (16) and the boundary values shown in Table 1 for given surface temperature distribution, and by (11), (17) and the boundary values shown in Table 2 for given surface heat flux distribution. In the figure, for simplicity's sake, only the values corresponding to ξ, $\xi^* = 0{\cdot}5$ are plotted.

Since both the cases of $n = 1$ in (5) and $m = 1$ in (10) correspond to the case of uniform heat transfer coefficient distribution, similarity solutions by Millsaps and Pohlhausen [5] can be transformed to the correlation terms in this paper. Details of this transformation are de-described in Appendix 2. The results are sumarized in Table 3, and inserted as two solid lines in Fig. 1.

Figure 1 exhibits that the correlation of $(Nu_x)_c/(Nu_x)_p$ vs. δ/r_w is scarcely affected by Prandtl number and by arbitrary constants N, M, n and m, which prescribe the shape of surface temperature or heat flux distribution. By eliminating ξ from (16), (6) and ξ^* from (17), (11), the following formulae are reduced respectively,

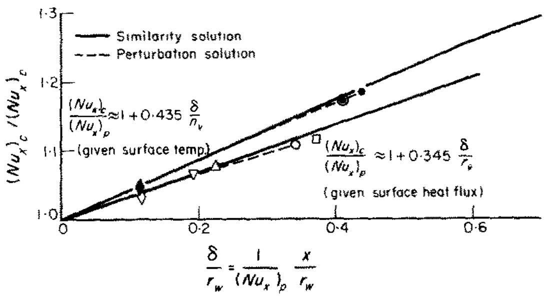

FIG. 1. $(Nu_x)_c/(Nu_x)_p$ the ratio of local Nusselt number for a cylinder to that for a flat plate vs. δ/r_w the ratio of boundary layer thickness for a flat plate to the radius of a cylinder. Symbols correspond to those in Tables 1, 2 and to $\xi, \xi^* = 0{\cdot}5$ respectively.

Table 2. Thermal boundary values for the case of uniform heat flux distribution, namely for $m = 0$ in (10)

Pr	$\theta_0^*(0)$	$\theta_1^*(0)$	$\theta_2^*(0)$	Symbols in Fig. 1
0·72	−1·4869	0·3710	−0·1140	□
1	−1·3574	0·3100	−0·0865	○
5	−0·9010	0·1396	−0·0270	△
10	−0·7675	0·1011	−0·0150	▽
100	−0·4656	0·0359	−0·0026	ˋ

Table 3. Relation of $(Nu_x)_c/(Nu_x)_p$ to δ/r_w for the case of uniform heat-transfer coefficient distribution

Pr	$H(0)$	Nu_r	Given temperature δ/r_w	Given temperature $\frac{(Nu_x)_c}{(Nu_x)_p}$	Given heat flux δ/r_w	Given heat flux $\frac{(Nu_x)_c}{(Nu_x)_p}$
	50	1·852	0·6991	1·295	0·6604	1·223
	100	2·127	0·5879	1·250	0·5611	1·194
	500	2·978	0·3931	1·171	0·3803	1·133
0·733	1000	3·461	0·3306	1·144	0·3214	1·112
	5000	4·960	0·2211	1·097	0·2167	1·075
	10000	5·818	0·1859	1·082	0·1829	1·064
	50000	8·479	0·1243	1·054	0·1227	1·040
	100000	10·000	0·1045	1·045	0·1037	1·037
	100	2·316	0·5310	1·230	0·5087	1·178
1	1000	3·794	0·2986	1·133	0·2922	1·109
	10000	6·407	0·1679	1·076	0·1652	1·058
	100000	11·05	0·0944	1·043	0·0936	1·034
	100	4·225	0·2656	1·122	0·2599	1·098
10	1000	7·164	0·1494	1·070	0·1476	1·058
	10000	12·38	0·0840	1·040	0·0834	1·032
	100000	21·65	0·0472	1·022	0·0470	1·018
	100	7·466	0·1445	1·079	0·1418	1·059
100	1000	12·91	0·0813	1·049	0·802	1·035
	10000	22·59	0·0457	1·032	0·0454	1·026
	100000	39·56	0·0257	1·016	0·0256	1·013

$$\frac{(Nu_x)_c}{(Nu_x)_p} = 1 - 2\theta_1'(0)\frac{\delta}{r_w} + 4\theta_0'(0)\,\theta_2'(0)\left(\frac{\delta}{r_w}\right)^2 + \ldots, \tag{18}$$

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \left[1 - \frac{2\theta_1^*(0)}{\theta_0^{*2}(0)}\frac{\delta}{r_w} + \frac{4\theta_2^*(0)}{\theta_0^{*3}(0)}\left(\frac{\delta}{r_w}\right)^2 + \ldots\right]^{-1}. \tag{19}$$

By substituting the boundary values corresponding to $n = 0$ and $Pr = 1$ as the representative for the temperature distribution indicated by (5) into (18), the following formula is obtained,

$$\frac{(Nu_x)_c}{(Nu_x)_p} = 1 + 0{\cdot}447\frac{\delta}{r_w} - 0{\cdot}057\left(\frac{\delta}{r_w}\right)^2 + \ldots \tag{20}$$

Similarly, from (19) the formula for $m = 0$ and $Pr = 1$ is obtained as

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \left[1 - 0{\cdot}336\frac{\delta}{r_w} + 0{\cdot}139\left(\frac{\delta}{r_w}\right)^2 + \ldots\right]^{-1}. \tag{21}$$

Formulae (20), (21) are inserted as two dotted lines in Fig. 1.

(iv) *Considerations*

The difference between the two Nu_x ratio formulae (20) and (21) is caused by the difference on the signification of $(Nu_x)_c/(Nu_x)_p$ between the cases of given surface temperature and heat flux. For the case of given surface temperature distribution,

because $$(t_w - t_\infty)_c = (t_w - t_\infty)_p, \tag{22}$$

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \frac{(\alpha_x)_c}{(\alpha_x)_p} = \frac{(q_x)_c}{(q_x)_p}. \tag{23}$$

On the other hand, for the case of given surface heat flux distribution, because

$$(q_x)_c = (q_x)_p, \tag{24}$$

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \frac{(\alpha_x)_c}{(\alpha_x)_p} = \frac{(t_w - t_\infty)_p}{(t_w - t_\infty)_c}. \tag{25}$$

The perturbation solutions inserted as dotted lines agree well with the similarity solutions inserted as solid lines in the range of smaller δ/r_w. In larger δ/r_w, the latter seems to be more accurate, because only two terms of perturbation are adopted in the former. The similarity solutions may be expressed approximately as follows;
for given surface temperature distribution,

$$(Nu_x)_c = (Nu_x)_p + 0{\cdot}435\frac{x}{r_w}, \qquad \frac{x}{r_w} \lesssim 0{\cdot}7\,(Nu_x)_p, \tag{26}$$

for given surface heat flux distribution,

$$(Nu_x)_c = (Nu_x)_p + 0{\cdot}345\frac{x}{r_w}, \qquad \frac{x}{r_w} \lesssim 0{\cdot}7\,(Nu_x)_p. \tag{27}$$

Although the ranges of ξ, ξ^* or δ/r_w applicable to (6), (11) or (18), (19) depend on Prandtl number as shown in Fig. 1 and Table 3, the ranges of x/r_w in (26), (27) may be extended at least up to 0·7 $(Nu_x)_p$ for any Prandtl number, since the analytical data on each distribution are uniquely correlated. By the way, in order to evaluate $(Nu_x)_c$ for a cylinder from (26) and (27), $(Nu_x)_p$ for a flat plate must be known. The method of approximate estimation of $(Nu_x)_p$ is shown in Appendix 3.

3. CONCLUSION

Heat-transfer coefficients on laminar natural convection in fluid of $Pr = 0{\cdot}72 \sim 100$ along the outer surface of a vertical cylinder, the temperature or heat flux distribution on which was given as a power function of the vertical distance, were treated. For these two cases, $(Nu_x)_c/(Nu_x)_p$ was uniquely correlated by δ/r_w respectively, where some data were calculated by perturbation method and some were referred from literature.

Practical approximate solutions are recommended by (26) and (27). These formulae are probably appreciable to the convection from a vertical cylinder with arbitrary vertical temperature or heat flux distribution to the fluid of any Prandtl number, because they are reduced on the basis of the thermal boundary layer thickness.

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APPENDIX 1

Perturbation equations

When the stream function ψ is introduced such that

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r}, \qquad v = -\frac{1}{r}\frac{\partial\psi}{\partial x}, \tag{28}$$

continuity equation (1) is automatically satisfied. Non-dimensional stream function and temperature profile in the boundary layer f, θ, are introduced respectively such that

$$\psi(x, r) = 2(\sqrt{2})\, \nu r_w\, Gr_x^{\frac{1}{4}} f(\eta, \xi), \tag{29}$$

$$\frac{t - t_\infty}{t_w - t_\infty} = \theta(\eta, \xi), \tag{30}$$

where

$$\eta = \frac{r^2 - r_w^2}{2(\sqrt{2})\, r_w x} Gr_x^{\frac{1}{4}}. \tag{31}$$

When $r_w \to \infty$, the independent variable η tends to

$$\frac{1}{\sqrt{2}} Gr^{\frac{1}{4}} \frac{r - r_w}{x}, \tag{32}$$

which is equal to the similarity variable for a flat plate [6]. By substituting (28)–(31) and (7) into (2)–(4), the following equations and boundary conditions are obtained,

$$(1 + \xi\eta)\frac{\partial^3 f}{\partial \eta^3} + \xi\frac{\partial^2 f}{\partial \eta^2} + (n - 1)\,\xi\left(\frac{\partial^2 f}{\partial\eta\partial\xi}\frac{\partial f}{\partial \eta} - \frac{\partial^2 f}{\partial \eta^2}\frac{\partial f}{\partial \xi}\right) + (n + 3) f\frac{\partial^2 f}{\partial \eta^2} - 2(n + 1)\left(\frac{\partial f}{\partial \eta}\right)^2 + \theta = 0, \tag{33}$$

$$(1 + \xi\eta)\frac{\partial^2 \theta}{\partial \eta^2} + \xi\frac{\partial \theta}{\partial \eta} + Pr\left[(n - 1)\,\xi\left(\frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \xi} - \frac{\partial \theta}{\partial \eta}\frac{\partial f}{\partial \xi}\right) + (n + 3) f\frac{\partial \theta}{\partial \eta} - 4n\theta\frac{\partial f}{\partial \eta}\right] = 0, \tag{34}$$

$$\left.\begin{aligned} &f = \frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial \xi} = 0, \quad \theta = 1, \quad &&\text{at} \quad \eta = 0, \\ &\frac{\partial f}{\partial \eta} = \theta = 0, &&\text{at} \quad \eta = \infty. \end{aligned}\right\} \tag{35}$$

When f and θ may be expressed as follows;

$$f(\eta, \xi) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots, \tag{36}$$

$$\theta(\eta, \xi) = \theta_0(\eta) + \xi\theta_1(\eta) + \xi^2\theta_2(\eta) + \ldots, \tag{37}$$

by substituting (36), (37) into (33), (34) and equating the coefficients of each power of ξ identically to zero, the following groups of ordinary differential equations are obtained,

$$\left.\begin{aligned} &f_0''' + (n + 3) f_0'' f_0 - (2n + 2)(f_0')^2 + \theta_0 = 0, \\ &\theta_0'' + Pr[(n + 3) f_0 \theta_0' - 4n f_0' \theta_0] = 0, \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} &f_1''' + (n + 3) f_0 f_1'' - (3n + 5) f_0' f_1' + 4 f_0'' f_1 + \theta_1 + \eta f_0''' + f_0'' = 0, \\ &\theta_1'' + (n + 3)\, Pr f_0 \theta_1' - (3n + 1)\, Pr f_0' \theta_1 - 4n Pr\, \theta_0 f_1' + 4 Pr \theta_0' f_1 + \eta\theta_0'' + \theta_0' = 0, \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} &f_2''' + (n + 3) f_0 f_2'' - 2(n + 3) f_0' f_2' - (n - 5) f_0'' f_2 + \theta_2 + \eta f_1''' + f_1'' - (n + 3)(f_1')^2 + 4 f_1'' f_1 = 0, \\ &\theta_2'' + (n + 3)\, Pr f_0 \theta_2' - 2(n + 1)\, Pr f_0' \theta_2 - 4n Pr \theta_0 f_2' - (n - 5)\, Pr \theta_0' f_2 + \eta\theta_1'' + \theta_0' - (3n + 1)\, Pr f_1' \theta_1 + 4 Pr f_1 \theta_1' = 0, \end{aligned}\right\} \tag{40}$$

. . . .

and boundary conditions are

$$\left.\begin{aligned} &f_0 = f_1 = f_2 = \ldots = 0 \\ &f_0' = f_1' = f_2' = \ldots = 0, \\ &\theta_0 = 1, \quad \theta_1 = \theta_2 = \ldots = 0, \end{aligned}\right\} \text{at} \quad \eta = 0, \tag{41}$$

$$\left.\begin{aligned} &f_0' = f_1' = f_2' = \ldots = 0, \\ &\theta_0 = \theta_1 = \theta_2 = \ldots = 0, \end{aligned}\right\} \text{at} \quad \eta = \infty, \tag{42}$$

where the primes denote differentiation with respect to η.

Equations (38) are non-linear simultaneous equations and equal to those reduced from similarity transformation of the boundary layer equations for a flat plate [6]. f_0, θ_0 and their derivatives, together with unknown boundary values $f_0''(0)$, $\theta_0'(0)$, are solved by trial and error computation. Since equations (39), (40) are linear, they are solved easily by superposition method.

Similarly, are reduced the ordinary differential equations for the case that the surface heat flux distribution is expressed by (10). In this case, stream function, non-dimensional temperature profile and independent variable are defined respectively by

$$\psi = 5^{\frac{4}{5}} \nu r_w Gr_x^{*\frac{1}{5}} f^*(\eta^*, \xi^*), \tag{29'}$$

$$f^*(\eta^*, \xi^*) = f_0(\eta^*) + \xi^* f_1^*(\eta^*) + \xi^{*2} f_2^*(\eta^*) + \ldots, \tag{36'}$$

$$\frac{\lambda(t_\infty - t)}{5^{\frac{1}{5}} q_w x} Gr_x^{*\frac{1}{5}} = \theta^*(\eta^*, \xi^*), \tag{30'}$$

$$\theta^*(\eta^*, \xi^*) = \theta_0^*(\eta^*) + \xi^*\theta_1^*(\eta^*) + \xi^{*2}\theta_2^*(\eta^*) + \ldots, \tag{37'}$$

$$\eta^* = \frac{r^2 - r_w^2}{2 \times 5^{\frac{1}{5}} r_w x} Gr_x^{*\frac{1}{5}}. \tag{31'}$$

The basic ordinary differential equations and boundary conditions, for example, are

$$\left.\begin{aligned} &f_0^{*'''} + (m + 4) f_0^* f_0^{*''} - (2m + 3)(f_0^{*'})^2 - \theta_0^* = 0, \\ &\theta_0^{*''} - Pr[(4m + 1)\,\theta_0^* f_0^{*'} - (m + 4) f_0^* \theta_0^{*'}] = 0, \end{aligned}\right\} \tag{38'}$$

$$f_0^* = f_0^{*'} = 0, \quad \theta_0^{*'} = 1, \quad \text{at} \quad \eta^* = 0, \tag{41'}$$

$$f_0^{*'} = \theta_0^* = 0, \quad \text{at} \quad \eta^* = \infty, \tag{42'}$$

where the primes denote differentiation with respect to η^*.

APPENDIX 2

Relation of $(Nu_x)_c/(Nu_x)_p$ to δ/r_w reduced from the similarity solutions by Millsaps and Pohlhausen

Similarity solutions of the boundary-layer equations along a vertical cylinder are obtainable only for the distribution of uniform heat-transfer coefficient, namely for $n = 1$ in (5) or $m = 1$ in (10). These solutions were obtained by Millsaps and Pohlhausen, and the Nusselt number Nu_r was represented by a parameter $H(0)$ corresponding to a thermal boundary value, where Nu_r and $H(0)$ are defined respectively by

$$Nu_r = \frac{\alpha r_w}{\lambda} = \frac{q_w r_w}{(t_w - t_\infty)\lambda}, \tag{43}$$

$$H(0) = \frac{g\beta r_w^4 (t_w - t_\infty)}{x v^2}. \tag{44}$$

$(Nu_x)_c$ and δ/r_w in this paper are expressed by these Nu_r and $H(0)$ as follows; corresponding to $n = 1$ in (5),

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \frac{\sqrt{2}\, Nu_r}{-\theta_0'(0)\,\{H(0)\}^{\frac{1}{4}}}, \tag{45}$$

$$\frac{\delta}{r_w} = \frac{\sqrt{2}}{-\theta_0'(0)\,\{H(0)\}^{\frac{1}{4}}}, \tag{46}$$

and corresponding to $m = 1$ in (10),

$$\frac{(Nu_x)_c}{(Nu_x)_p} = \frac{-5^{\frac{1}{5}}\theta_0^*(0)\, Nu_r^{\frac{4}{5}}}{\{H(0)\}^{\frac{1}{5}}}, \tag{47}$$

$$\frac{\delta}{r_w} = \frac{-5^{\frac{1}{5}}\theta_0^*(0)}{\{Nu_r H(0)\}^{\frac{1}{5}}}, \tag{48}$$

where $\theta_0'(0)$ and $\theta_0^*(0)$ are thermal boundary values for a flat plate with the same surface temperature or heat flux distribution.

These values of $\theta_0'(0)$ and $\theta_0^*(0)$ are exterpolated from the numerical values of Nu_r and $H(0)$ by means of the following physical relation respectively,

$$\frac{Nu_r}{\{H(0)\}^{\frac{1}{4}}} \to \frac{-\theta_0'(0)}{\sqrt{2}} \quad \text{at} \quad \frac{1}{\{H(0)\}^{\frac{1}{4}}} \to 0,$$

$$\frac{Nu_r^{\frac{4}{5}}}{\{H(0)\}^{\frac{1}{5}}} \to \frac{5^{\frac{1}{5}}}{-\theta_0^*(0)} \quad \text{at} \quad \frac{1}{\{Nu_r H(0)\}^{\frac{1}{5}}} \to 0$$

instead of direct solving the original differential equations. The results are shown in Table 4.

By (45), (46) and $\theta_0'(0)$ or (47), (48) and $\theta_0^*(0)$, the relation of $(Nu_x)_c/(Nu_x)_p$ to δ/r_w are reckoned as shown in Table 3.

Table 4. Thermal boundary values for the case of uniform heat transfer coefficient distribution on a flat plate

Pr	0·733	1	10	100
$-\theta_0'(0)$	0·7607	0·8422	1·6839	3·0954
$-\theta_0^*(0)$	1·1905	1·0971	0·6304	0·3871

APPENDIX 3

Local Nusselt number for a flat plate, the temperature or heat flux distribution on which is given by (5) or (10) respectively

The local Nusselt numbers are determined by the thermal boundary values of (38) and (38)′ as shown in (8) and (13). When the surface temperature distribution is given by (5), the surface heat flux distribution q_w is obtained as

$$q_w = -2^{\frac{1}{4}} N\theta_0'(0)\,\lambda Gr_x^{\frac{1}{4}} x^{(5n-1)/4}, \tag{49}$$

and when the surface heat flux distribution is given by (10), the surface temperature distribution is

$$t_w - t_\infty = -5^{\frac{1}{5}} M\theta_0^*(0)\,\lambda^{-1} Gr_x^{*\,-\frac{1}{5}} x^{(4m+1)/5}, \tag{50}$$

where $\theta_0'(0)$ and $\theta_0^*(0)$ are the same thermal boundary values with those shown in Tables 1, 2 and 4.

Since each of the obtained heat flux and temperature distribution (49), (50) is a power function of x, these two cases must be physically identical. By equating (5) to (50) and (10) to (49) respectively, following relations are reduced,

$$4m = 5n - 1. \tag{51}$$

$$-\theta_0^*(0) = (\tfrac{4}{5})^{\frac{1}{5}}\{-\theta_0'(0)\}^{-\frac{4}{5}}. \tag{52}$$

If either $\theta_0'(0)$ for a value of n or $\theta_0^*(0)$ for a value of m is solved, the other for the value of m or n exchanged by (51) is obtained by (52). Since the boundary values for $n = 0$, say $[\theta_0'(0)]_{n=0}$, were computed precisely by Ostrach [7] for $Pr = 0{\cdot}01$, 0·72, 0·733, 1, 2, 10, 100 and 1000, a plot of the ratio $\theta_0'(0)/[\theta_0'(0)]_{n=0}$ vs. n is attempted as shown in Fig. 2. This figure exhibits that the boundary value ratios are scarcely affected by Prandtl number. As to the value for $n = 1$ and $Pr = 100$, the convergency of the numerical data at $\eta \to \infty$ is somewhat doubtful.

For more precise evaluation, equation (38) must be solved numerically. Then $\theta_0'(0)$ shown in Fig. 2 and $f_0''(0)$ given by the following formula are recommended as the starting values for the trial and error computation.

$$f_0''(0) \approx 0{\cdot}475 - 0{\cdot}667 \log \theta_0'(0), \qquad 0{\cdot}72 < Pr < 100.$$

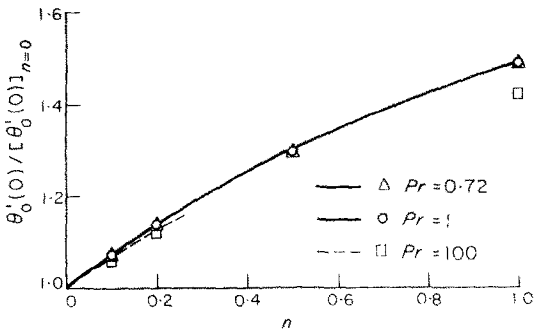

FIG. 2. $\theta_0'(0)/[\theta_0'(0)]_{n=0}$ the ratio of the thermal boundary value for arbitrary exponent n in (5) to that for $n = 0$ vs. n, with respect to a flat plate. Data on $n = 0{\cdot}1$ were computed by authors and those on $n = 0{\cdot}5$ were referred from [6].

TRANSPORT DE CHALEUR PAR CONVECTION NATURELLE LAMINAIRE À PARTIR DE LA SURFACE EXTÉRIEURE D'UN CYLINDRE VERTICAL

Résumé—La convection naturelle laminaire le long de la surface extérieure d'un cylindre vertical est comparée, au point de vue transport de chaleur, avec celle le long d'une plaque plane verticale. Pour n'importe quel nombre de Prandtl et pour une distribution verticale arbitraire de flux de chaleur ou de température à la surface du cylindre, les coefficients de transport de chaleur local sont représentés de façon adimensionnelle par les formules approchées suivantes.

$$(Nu_x)_c - (Nu_x)_p = 0{,}435\frac{x}{r_w}, \qquad \frac{x}{r_w} \lesssim 0{,}7\,(Nu_x)_p$$

et lorsque la distribution de flux de chaleur à la surface est donnée,

$$(Nu_x)_c - (Nu_x)_p = 0{,}345\frac{x}{r_w}, \qquad \frac{x}{r_w} \lesssim 0{,}7\,(Nu_x)_p$$

où $(Nu_x)_c$ et $(Nu_x)_p$ sont respectivement les nombres de Nusselt locaux pour un cylindre et une plaque plane, x la distance verticale à partir du bord d'attaque et r le rayon du cylindre.

WÄRMEÜBERGANG BEI LAMINARER FREIER KONVEKTION AN DER AUSSENFLÄCHE EINES SENKRECHTEN ZYLINDERS

Zusammenfassung—Der Wärmeübergang bei laminarer freier Konvektion an der Aussenfläche eines senkrechten Zylinders wird mit dem an einer senkrechten ebenen Platte verglichen. Für beliebige Prandtl-Zahlen und für beliebige Temperatur- oder Wärmestromverteilung längs der Zylinderoberfläche lassen sich die örtlichen Wärmeübergangskennzahlen dimensionslos darstellen nach folgenden Näherungsformeln.

Bei gegebener Oberflächentemperaturverteilung

$$(Nu_x)_c - (Nu_x)_p = 0{,}435\frac{x}{r_w}; \qquad \frac{x}{r_w} \lesssim 0{,}7\,(Nu);$$

und bei gegebener Wärmestromverteilung

$$(Nu_x)_c - (Nu_x)_p = 0{,}345\frac{x}{r_w}; \qquad \frac{x}{r_w} \lesssim 0{,}7\,(Nu);$$

Dabei sind $(Nu_x)_c$ und $(Nu_x)_p$ die örtliche Nusseltzahlen für Zylinder bzw. Platte, x der senkrechte Abstand von Vorderkante und r der Radius des Zylinders.

ТЕПЛООБМЕН ВНЕШНЕЙ ПОВЕРХНОСТИ ВЕРТИКАЛЬНОГО ЦИЛИНДРА ПРИ ЛАМИНАРНОЙ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ

Аннотация—Сравнивается теплообмен вертикального цилиндра при естественной ламинарной конвекции с теплообменом вертикальной плоской пластины, помещенной в аналогичных условиях. Представлены в безразмерном виде аппроксимирующие формулы для расчета локальных коэффициентов теплообмена для любых значений числа Прандтля произвольного распределения температуры или теплового потока по поверхности цилиндра.

а) задано распределение температуры на поверхности

$$(Nu_x)_c = (Nu_x)_p + 0{,}435\frac{x}{r_w}, \; \frac{x}{r_w} \leqslant 0{,}7(Nu_x)_p$$

б) задано распределение теплового потока по поверхности

$$(Nu_x)_c = (Nu_x)_p + 0{,}375\frac{x}{r_w}, \; \frac{x}{r_w} \leqslant 0{,}7(Nu_x)_p$$

здесь $(Nu_x)_c$ и $(Nu_x)_p$—локальные числа Нуссельта для цилиндра и плоской пластины, соответственноі x-расстояние по вертикали от передней кромкиі r_w-радиус цилиндра.